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A NOVEL DESIGN FOR A HYBRID SPACE MANIPULATOR

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ABSTRACT:

Described are the structural design, kinematics and characteristics of a novel robotic manipulator for space applications and, in particular, utilization as an articulate and powerful space shuttle manipulator. Hybrid manipulators are parallel–serial connection robots that give rise to a multitude of highly articulate robot manipulators. These manipulators are modular and can be extended by additional modules over large distances. Every module has a hemi–spherical work space and collective modules give rise to highly dexterous symmetrical work space.In this paper some basic designs and kinematical structures of these robot manipulators are discussed, the associated direct and the inverse kinematics formulations are presented, and solutions to the inverse kinematic problem are obtained explicitly and elaborated upon.

These robot manipulators are shown to have a strength–to–weight ratio that is many times larger than the value that is currently available with industrial or research manipulators. This is due to the fact that these hybrid manipulators are stress–compensated and have an ultra light weight, yet, they are extremely stiff due to the fact that the force distribution in their structure is mostly axial. The means of actuation in these manipulators are entirely prismatic and can be provided by ball–screws with anti–backlash nuts for maximum precision.

INTRODUCTION

Serially connected robot manipulators in the form of an open–loop kinematic chain with computer–controlled joint actuation have been examined extensively in the robot engineering literature (see Shahinpoor [1]). These examinations study the structural design, kinematics, dynamics, trajectory planning, work space design, control and stability. On the other hand the pertinent literature on parallel–connection robot manipulators is scarce as discussed by Fichter [2][3]. A classic example of a parallel manipulator is the Stewart platform (see Stewart [4]) which has been kinematically and to some extent dynamically investigated by Fichter [2]. Other similar mechanisms and manipulators have been discussed by Earl and Rooney [5], Hunt [6], and Yang and Lee [7].

In the present paper we introduce yet another novel robotic structure of a hybrid nature. In these hybrid manupulators both serial elements and parallel elements are present and can be actuated in a prismatic fashion to give rise to a highly articulate robot manipulator with hemispherical work space and complete symmetry of movements within its work space. Figure 1 illustrates such a hybrid robot manipulator. Note that this structure particularly relates to a computer–controlled robotic arm capable of moving three dimensionally and symmetrically throughout its hemispherical workspace.

Computer–controlled robotic arms have been extensively used throughout the world and particularly in the US and Japan. See Shahinpoor [1] for a comprehensive literature survey on various kinds of robot manipulators and structural designs. Two basic problems have been associated with conventional robot manipulators as described below:

1– They are generally made massive and stiff so as to eliminate motion control problems associated with structural flexibility.

2– They generally move slow because of the fact that they are made massive and faily rigid.

Thus, there has been a great need in the manufacturing industry, government laboratories as well as defense organizations to develop light–weight, stiff and subsequently fast moving robot manipulators. The structure shown in Figure 1 and described in the following section achieves the above objectives and corrects for the above deficiencies of the conventional robot manipulators. Since all of its legs are simply supported at both ends by three dimensional joints such as universal or ball–and–socket joints the stresses in them are only axially distributed and thus give rise to a stress–compensated robotic structures. The structure shown in Figure 1 also has a minimum amount of extra mass and is essentially an ultra–light weight manipulator. Thus, it provides an ultra–light weight, stress–compensated robotic arm capable of fast motions. It is further capable of moving symmetrically and hemi–spherically about its base platform ; something that most current robotic structures are unable to do.

In accordance with the present paper we describe a 7 degree of freedom robotic arm comprising a three–dimensional universal joint and two segments of a robotic arm such that the one end of the first segment is fixed to a base platform in the form of an equilateral triangular structure with the other end attaced to a joint platform which is another equilateral triangular structure. The one end of the second segment is attached to the joint platform with the other end attached to a gripper platform which is another equilateral triangular structure such that it is basically free to move but other–wise equipped with a robotic hand, gripper, end–effector or fixture. The base platform is comprised of an equilateral triangle whose sides are made from metallic or otherwise strong material. The said joint platform is comprised of another equilateral triangular structure with strong sides positioned opposite to the base platform such that the vertices of the base platform and the joint platform are connected by means of a triplet of criss–crossed woven or single wires with a movable joint. The gripper platform is also an equilateral trianglular structure positioned oppositely to the sides of the joint platform such that the vertices are connected oppositely to the vertices of the joint platform first by a pair of criss–crossed or woven wires and also to the middle points of the sides of the joint platform by means of a set of linear actuator. The end–effector which may be a robotic gripper is attached via a set of support bars to yet another equilateral triangle, namely, an extended gripper platform with sides and vertices oppositely oriented with respect to the gripper platform. The gripper action may be provided by an intermediate mechanism.

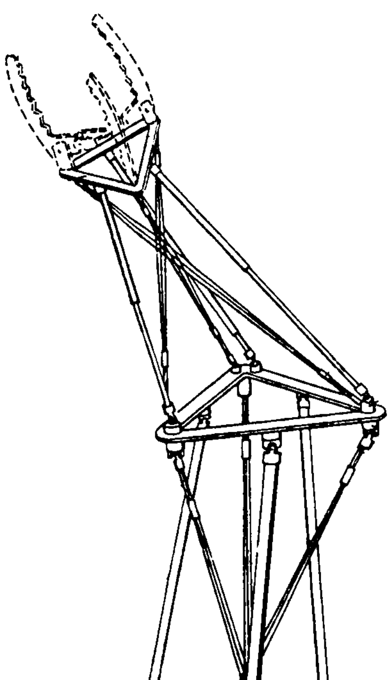

Figure 1– A platform structure of a hybrid robot manipulator

The actuation is provided by a set of six linear actuators . These linear actuators are such that three of them connect the vertices of the base platform to the mid section of the sides of the joint platform. Subsequently the other three linear actuators connect the mid section of the sides the joint platform to the vertices of the gripper platform.

The linear actuation may be hydraulic, pneumatic or electromagnetic. In case of hydraulic or pneumatic actuation the fluid motion control is provided by either digital or analog controllers comprising of electromagnetic valves. In case of electromagnetic actuation the linear actuators may be magnetic–induction or magnetic–coil driven or comprised of motorized ball screws for linear actuation. The gripper may also be actuated either hydraulically, pneumatically or electromagnetically. Due to the fact that the support bars create a kinematically constrained motion for the platforms the linear motion of the actuators must be performed in harmony so as not to violate the kinematical constraints. Here below a complete kinematic description of this robot manipulator is presented. This kinematical modeling is necessary for computer–controlled motion of the robotic gripper.

The fundamental question answered here is :

"Given the desired location and orientation of the gripper in the hemi–spherical work space of the manipulator what are the six values of the linear displacements of the 6 actuators in order to place the said gripper correctly at the desired position and with the desired orientation."

Let us refer now to Figure 2 which depicts a kinematic embodiment of the invention.

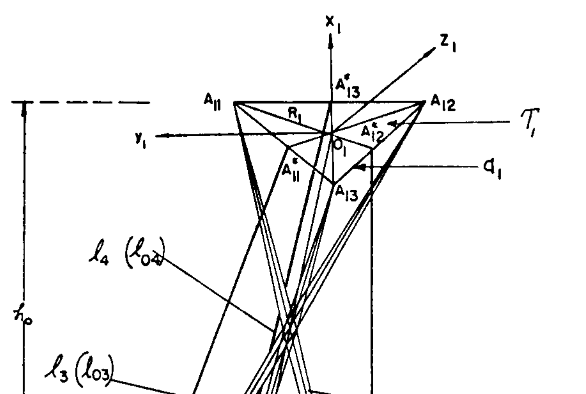

Figure 2– The kinematical structure of a 3–axis hybrid manipulator.

Note that, the present an analytical representation of the kinematic of the hybrid manipulator, a fixed reference rectangular cartesian frame is assumed with its origin at the point of intersection of angle bisectors or the medians of the said base platform, which is , hereon, called T_0 A corresponding rectangular cartesian frame x, y, z, is considered fixed to the center of the said joint platform which is called T_1 .

The locations of points A_{01}, A_{02}, A_{03}, A_{11} A_{12} and A_{13} are given by

$$R_{A_{01}} = ((\sqrt{3}/6)a_0, -(a_0/2), 0)^T \quad (1)$$

$$R_{A_{02}} = ((\sqrt{3}/6)a_0, (1/2)a_0, 0)^T \quad (2)$$

$$R_{A_{03}} = ((-\sqrt{3}/3)a_0, 0 , 0)^T \quad (3)$$

with respect to the base frame T_0 and by

$$R_{A_{11}} = ((+\sqrt{3}/6)a_1, +(1/2)a_1, 0)^T \quad (4)$$

$$R_{A_{12}} = ((3/6)a_1, -(1/2)a_1, 0)^T \quad (5)$$

$$R_{A_{13}} = ((-\sqrt{3}/6)a_1, 0, 0)^T \quad (6)$$

with respect to the platform frame T_1.

Consider an equilibrium reference position of the upper triangle T_1 with respect to the lower triangle T_0 such that they are parallel with a perpendicular separation of h_0 for which all lengths l_1 through l_6 are equal to l_1 through l_6. Under these circumstances the position of O_1 the origin of the frame T_1 with respect to T_0 is given by a vector $\underset{\sim}{r}_0$ which is, however, generally $\underset{\sim}{r}$.

In the reference configuration the coordinate frame T_1 can be expressed with respect to the frame T_0 by means of a 4x4 homogeneous transformation

$$[T_1]_0 = \begin{bmatrix} -1 & 0 & 0 & -b_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

Now let the origin of the T_1 frame in the upper said joint platform move to an orbitrary position $\underset{\sim}{r} = (r_x, \underset{\sim}{r}_y, r_z)^T$ and orientation θ, φ, ψ, such that θ, φ, and ψ are the corresponding angles in a right–handed fashion, between the pairs of axes (x_0, x_1), (y_0, y_1), and (z_0, z_1), respectively. In this arbitrary position and orientation the frame T_1 can be expressed with respect to the frame T_0 by means of another 4x4 homogeneous transformation $[T_1]$ such that

$$[T_1] = \begin{bmatrix} \cos\theta & \cos(x_1,y_0) & \cos(x_1,z_0) & r_x \\ \cos(y_1,x_0) & \cos\varphi & \cos(y_1,z) & r_y \\ \cos(z_1,x_0) & \cos(z_1,y_0) & \cos\psi & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (8)$$

or

$$[T_1] = \begin{bmatrix} d_{11} & d_{12} & d_{13} & r_x \\ d_{21} & d_{22} & d_{23} & r_y \\ d_{31} & d_{32} & d_{33} & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

where d_{ij}, i,j = 1,2,3 are the direction cosines between the T_0 and the T_1 frames, i.e.,

$$d_{ij} = \mathrm{Cos}\,(x_{1i}, x_{jo}), \quad (10)$$

$$x_{1i} = (x_1, y_1, z_1)^T, \quad (11)$$

$$(x_{jo}) = (x_o, y_o, z_o)^T \quad (12)$$

Thus, the location of all points on the upper triangle can be obtained with respect to the T_1 frame such that

$$0_1 \rightarrow \quad r_{0_1} = (0,0,0)^T \quad (13)$$

$$A_{11} \rightarrow \quad R_{A_{11}} = ((\sqrt{3}/6)a_1, (1/2)a_1, 0)^T \quad (14)$$

$$A_{12} \rightarrow \quad R_{A_{12}} = (\sqrt{3}/6)a_1, -(1/2)a_1, 0)^T \quad (15)$$

$$A_{13} \rightarrow \quad R_{A_{13}} (-(\sqrt{3}/3)a_1, 0, 0)^T \quad (16)$$

$$A^*_{11} \rightarrow \quad R^*_{A_{11}} (-(\sqrt{3}/12)a_1, (1/4)a_1, 0)^T \quad (17)$$

$$A^*_{12} \rightarrow \quad R^*_{A_{12}} (-(\sqrt{3}/12)a_1, -(1/4)a_1, 0)^T \quad (18)$$

$$A^*_{13} \rightarrow \quad R^*_{A_{13}} ((\sqrt{3}/6)a_1, 0, 0)^T \quad (19)$$

and

$$
r_{A_{11}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A_{11}} = [T]\, R^{(H)}_{A_{11}} = [T] \begin{bmatrix} (\sqrt{3}/6)a_1 \\ (1/2)a_1 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} (\sqrt{3}/6)a_1 d_{11} + (1/2)a_1 d_{12} + r_x \\ (\sqrt{3}/6)a_1 d_{21} + (1/2)a_1 d_{22} + r_y \\ (\sqrt{3}/6)a_1 d_{31} + (1/2)a_1 d_{32} + r_z \\ 1 \end{bmatrix} \quad (20)
$$

where $R^{(H)}_{A_{11}}$ is the homogeneous representation of $R_{A_{11}}$. and Similary

$$
r_{A_{12}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A_{12}} = [T]\mathbf{R}^{(H)}_{A\ 12} = \begin{bmatrix} (\sqrt{3}/6)a \\ -(1/2)a_1 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} d_{11}\ (3/6)a_1 - (1/2)a_1 d_{12} + r_x \\ (\sqrt{3}/6)a_1 d_{21} - (1/2)a_1 d_{22} + r_y \\ (\sqrt{3}/6)a_1 d_{31} - (1/2)a_1 d_{32} + r_z \\ 1 \end{bmatrix} \quad (21)
$$

$$
r_{A_{13}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A_{13}} = [T_1]\, \mathbf{R}^{H}_{A_{13}} = [T_1] \begin{bmatrix} -(\sqrt{3}/3)a_1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -(\sqrt{3}/3)a_1 d_{11} + r_x \\ -(\sqrt{3}/3)a_1 d_{21} + r_y \\ -(\sqrt{3}/3)a_1 d_{31} + r_z \\ 1 \end{bmatrix} \quad (22)
$$

$$
r^*_{A_{11}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A^*_{11}} = [T_1]\mathbf{R}^{*H}_{A_{11}} = [T_1] = \begin{bmatrix} -(\sqrt{3}/12)a_1 \\ (1/4)a_1 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -(\sqrt{3}/12)a_1 d_{11} + (1/4)a_1 d_{12} + r_x \\ -(\sqrt{3}/12)a_1 d_{21} + (1/4)a_1 d_{22} + r_y \\ -(\sqrt{3}/12)a_1 d_{31} + (1/4)a_1 d_{32} + r_z \\ 1 \end{bmatrix} \quad (23)
$$

$$
r^*_{A_{12}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A^*_{12}} = [T_1]\mathbf{R}_{A_{12}}^{*H}\, [T_1] \begin{bmatrix} -(\sqrt{3}/12)a_1 \\ -(1/4)a_1 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -(\sqrt{3}/12)a_1 d_{11} - (1/4)a_1 d_{12} + r_x \\ -(\sqrt{3}/12)a_1 d_{21} - (1/4)a_1 d_{22} + r_y \\ -(\sqrt{3}/12)a_1 d_{31} - (1/4)a_1 d_{32} + r_z \\ 1 \end{bmatrix} \quad (24)
$$

$$
r^*_{A_{13}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A^*_{13}} = [T_1]\mathbf{R}^{H}_{A_{23}} = [T_1] \begin{bmatrix} (\sqrt{3}/6)a_1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} (\sqrt{3}/6)a_1 d_{11} + r_x \\ (\sqrt{3}/6)a_1 d_{21} + r_y \\ (\sqrt{3}/6)a_1 d_{31} + r_z \\ 1 \end{bmatrix} \quad (25)
$$

Note that,

$$
\ell_i^2 = (x_{A_{i2}} - x_{A_{o1}})^2 + (y_{A_{i2}} - y_{A_{o1}})^2 + (z_{A_{i2}} - z_{A_{o1}})^2, \quad \text{for } i = 1.2..6, \quad (26)
$$

or

$$
\ell_1^2 = ((\sqrt{3}/6)a_1 d_{11} - (1/2)a_1 d_{12} + r_x - (\sqrt{3}/6)a_o)^2 + ((\sqrt{3}/6)a_1 d_{21} - (1/2)a_1 d_{22} + r_y + (a_o/2))^2 + ((\sqrt{3}/6)a_1 d_{31} - (1/2)a_1 d_{32} + r_z) \quad (27)
$$

$$
\ell_2^2 = (x_{A^*_{11}} - x_{A_{o1}})^2 + (y_{A^*_{11}} - y_{A_{o1}})^2 + (z_{A^*_{11}} - z_{A_{o1}})^2 \quad (28)
$$

$$
\ell_3^2 = (x_{A_{13}} - x_{A_{o3}})^2 + (y_{A_{13}} - y_{A_{o3}})^2 + (z_{A_{13}} - z_{A_{o3}})^2 \quad (29)
$$

$$
\ell_4^2 = (x_{A^*_{13}} - x_{A_{o3}})^2 + (y_{A^*_{13}} - y_{A_{o3}})^2 + (z_{A^*_{13}} - z_{A_{o3}})^2 \quad (30)
$$

$$
\ell_5^2 = (x_{A_{11}} - x_{A_{o2}})^2 + (y_{A_{11}} - y_{A_{o2}})^2 + (z_{A_{11}} - z_{A_{o2}})^2 \quad (31)
$$

$$
\ell_6^2 = (x_{A^*_{12}} - x_{A_{o2}})^2 + (y_{A^*_{12}} - y_{A_{o2}})^2 + (z_{A^*_{12}} - z_{A_{o2}})^2 \quad (32)
$$

$$
\ell_2^2 = (-(\sqrt{3}/12)a_1 d_{11} + (1/4)a_1 d_{12} + r_x - (\sqrt{3}/6)a_o)^2 + (-(\sqrt{3}/6)a_1 d_{21} + (1/4)a_1 d_{22} + r_y + (a_o/2))^2 + (-(\sqrt{3}/6)a_1 d_{31} + (1/4)a_1 d_{32} + r_z)^2 \quad (33)
$$

$$\ell_3^2=(-(\sqrt{3}/3)a_1d_{11}+r_x+(\sqrt{3}/3)a_0)^2+(-(\sqrt{3}/3)a_1d_{21}+r_y)^2+(-(\sqrt{3}/3)a_1d_{31}+r_z)^2 \tag{34}$$

$$\ell_4^2=((\sqrt{3}/6)a_1d_{11}+r_x+(\sqrt{3}/3)a_0)^2+((\sqrt{3}/6)a_1d_{21}+r_y)^2+((\sqrt{3}/6)a_1d_{31}+r_z)^2 \tag{35}$$

$$\ell_5^2=((\sqrt{3}/6)a_1d_{11}+(1/2)a_1d_{12}+r_x-(\sqrt{3}/6)a_0)^2+((\sqrt{3}/6)a_1d_{21}+(1/2)a_1d_{22}+r_y-(1/2)a_0)^2+((\sqrt{3}/6)a_1d_{31}+(1/2)a_1d_{32}+r_z)^2 \tag{36}$$

$$\ell_6^2=(-(\sqrt{3}/12)a_1d_{11}-(1/4)a_1d_{12}+r_x-(\sqrt{3}/6)a_0)^2+((\sqrt{3}/12)a_1d_{21}-(1/4)a_1d_{22}+r_y-(1/2)a_0)^2+(-(\sqrt{3}/12)a_1d_{31}-(1/4)a_1d_{32}+r_z)^2 \tag{37}$$

Equations (26)–(37) represent a set of equations for the solution of the inverse kinematics problem of such a robot manipulator.

Note that given a desired position of the origin of the T_1 frame in the upper said joint platform, i.e., r_x, r_y, r_z, and a desired orientation of it with respect to the base frame T_0, i.e., θ, φ and ψ, the desired leg lengths ℓ_i, i=1,2,6 can be explicity determined. These ℓ_i's would then determine the extent of computer–controlled prismatic extension of the robot legs.

In our case the lengths ℓ_1, ℓ_3, and ℓ_5 are fixed and basiclly equal to some length ℓ_0. This means that equations (32), (34) and (36) now completely define the boundaries of the work space of the robot and equations (33), (35) and (37) can be used to determine the actuation lengths necessary to generate the desired attitude (position + orientation) of the upper platform. Furthermore, equations (32), (34), and (36) determine the values r_x, r_y and r_z as a function of θ, φ, and ψ given that ℓ_1, ℓ_3 and ℓ_5 are prescribed. Therefore, given the values of ℓ_1, ℓ_3 and ℓ_5 and the desired orientation of the frame T_1 with respect to T_0, equations (32)–(37) completely define an algorithm to achieve computer–controlled positioning of the first platform.

An exact similar analysis could be presented for the kinematics and the solution to the inverse kinematics problem of the second and if desired, the third platforms, respectively.

Extension To Multiple Platforms

Referring to Figure 3 below, we note that one may use a similar treatment for the frame T_2 or the said gripper platform with respect to the frames T_1 and T_0.

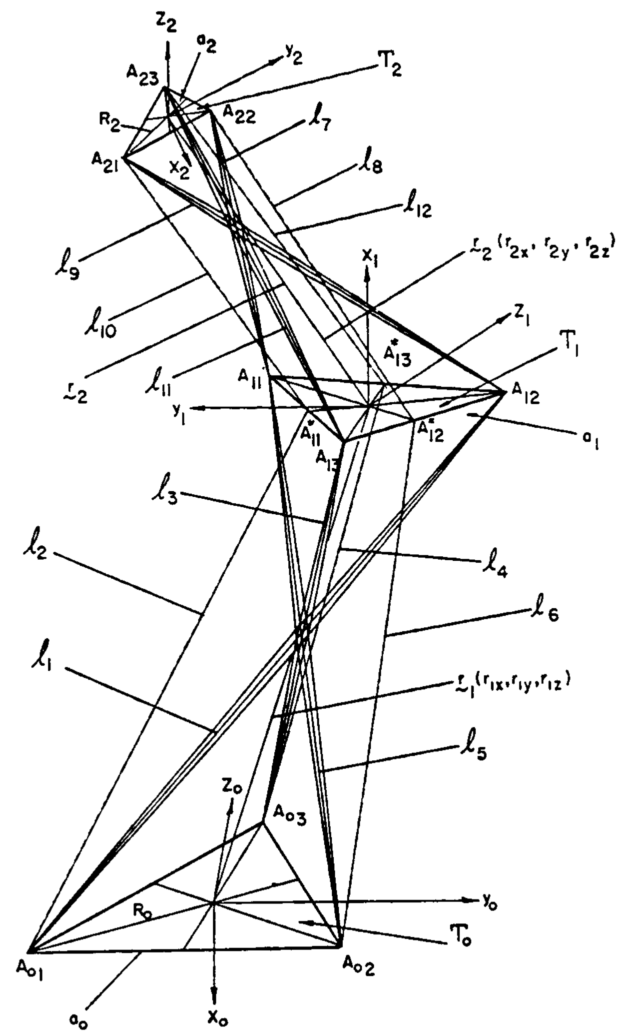

Figure 3– Kinematic structure of a 6–axis hybrid manipulator

Note that in this case

$$\ell_7^2= (x_{A_{22}}-x_{A_{11}})^2+(y_{A_{22}}-y_{A_{11}})^2+(z_{A_{22}}-z_{A_{11}})^2, \tag{38}$$

$$\ell_8^2= (x_{A_{22}}-x_{A^*_{12}})^2+(y_{A_{22}}-y_{A^*_{12}})^2+(z_{A_{22}}-z_{A^*_{12}})^2, \tag{39}$$

$$\ell_9^2=(x_{A_{21}}-x_{A_{12}})^2+(y_{A_{21}}-y_{12})^2+(z_{A_{21}}-z_{A^*_{12}})^2, \tag{40}$$

$$\ell_{10}^2=(x_{A_{21}}-x_{A^*_{11}})^2+(y_{A_{21}}-y_{A^*_{11}})^2+(z_{A_{21}}-z_{A^*_{11}})^2, \tag{41}$$

$$\ell_{11}^2=(x_{A_{23}}-x_{A_{13}})^2+(y_{A_{23}}-y_{A_{13}})^2+(z_{A_{23}}-z_{A_{13}})^2, \tag{42}$$

$$\ell_{12}^2=(x_{A_{23}}-x_{A^*_{13}})^2+(y_{A_{23}}-y_{A^*_{13}})^2+(z_{A_{23}}-z_{A^*_{13}})^2, \tag{43}$$

$$
r_{A_{21}}=\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A_{21}} = T_o^2\, R^H_{A_{21}} = T_o^2 \begin{bmatrix} (\sqrt{3}/6)a_2 \\ -(a_2/2) \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} (\sqrt{3}/6)a_2 d^*_{11} - (1/2) d^*_{12} + r_x \\ (\sqrt{3}/6)a_2 d^*_{21} - (1/2)a_2 d^*_{32} + r_y \\ (\sqrt{3}/6)a_2 d_{31} - (1/2)a_2 d_{22} + r_z \\ \end{bmatrix} \tag{44}
$$

$$
r_{A_{22}}=\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A_{22}} = T_o^2\, R^H_{A_{22}} = T_o^2 \begin{bmatrix} (\sqrt{3}/6)a_2 \\ (a_2/2) \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} (\sqrt{3}/6)a_2 d^*_{11} + (1/2)a_2 d^*_{12} + r_x \\ (\sqrt{3}/6)a_2 d^*_{21} + (1/2)a_2 d^*_{32} + r_y \\ (\sqrt{3}/6)a_2 d_{31} + (1/2)a_2 d_{22} + r_z \\ \end{bmatrix} \tag{45}
$$

$$
r_{A_{23}}=\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{A_{23}} = T_o^2\, R^H_{A_{23}} = T_o^2 \begin{bmatrix} -(\sqrt{3}/3)a_2 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -(\sqrt{3}/3)a_2 d^*_{11} + r_x \\ -(\sqrt{3}/3)a_2 d^*_{21} + r_y \\ -(\sqrt{3}/3)a_2 d_{31} + r_z \\ 1 \end{bmatrix} \tag{46}
$$

where d^*_{ij} are the direction cosines in T_o^2 transformations. Thus

$$
\ell_7^2=((\sqrt{3}/6)a_2 d^*_{11}-(1/2)a_2 d^*_{12}+r_x-(\sqrt{3}/6)a_1 d_{11}
-(1/2)a_1 d_{12}-r_{1x})^2+((\sqrt{3}/6)a_2 d^*_{21}-(1/2)a_2 d_{32}
+r_y-(\sqrt{3}/6)a_1 d_{21}-(1/2)a_1 d_{22}-r_{1y})^2+((\sqrt{3}/6)a_2 d^*_{31}
-(1/2)a_2 d^*_{22}+r_z-(\sqrt{3}/6)a_1 d_{31}-(1/2\_a_1 d_{32}-r_{1z})^2 \tag{47}
$$

$$
\ell_8^2=((\sqrt{3}/6)a_2 d^*_{11}+(1/2)a_2 d^*_{12}+r_x+(\sqrt{3}/12)a_1 d_{11}
+(1/4)a_1 d_{12}-r_{1x})^2+((\sqrt{3}/6)a_2 d^*_{21}+(1/2)a_2 d^*_{32}
+r_y+(\sqrt{3}/12)a_1 d_{21}+(1/4)a_1 d_{22}-r_{1y})^2
+((\sqrt{3}/6)a_2 d^*_{31}+(1/2)a_2 d^*_{22}+r_z+(\sqrt{3}/12)a_1 d_{31}
+(1/4)a_1 d_{32}-r_{1z})^2 \tag{48}
$$

$$
\ell_9^2=((\sqrt{3}/6)a_2 d^*_{11}-(1/2)a_2 d^*_{12}+r_x-(\sqrt{3}/6)a_1 d_{11}
+(1/2)a_1 d_{12}-r_{1x})^2+((\sqrt{3}/6)a_2 d^*_{21}
-(1/2)a_2 d^*_{32}+r_y-(\sqrt{3}/6)a_1 d_{21}
+(1/2)a_1 d_{22}-r_{1y})^2+((\sqrt{3}/6)a_2 d^*_{31}
-(1/2)a_2 d^*_{22}+r_z-(\sqrt{3}/6)a_1 d_{31}
+(1/2)a_1 d_{32}-r_{1z})^2 \tag{49}
$$

$$
\ell_{10}^2=((\sqrt{3}/6)a_2 d^*_{11}-(1/2)a_2 d^*_{12}+r_x
+(\sqrt{3}/12)a_1 d_{11}-(1/4)a_1 d_{12}-r_{1x})^2
+((\sqrt{3}/6)a_2 d^*_{21}-(1/2)a_2 d^*_{32}+r_y+(\sqrt{3}/12)a_1 d_{21}
-(1/4)a_1 d_{22}-r_{1y})^2+((\sqrt{3}/6)a_1 d^*_{31}
-(1/2)a_2 d^*_{22}+r_z+(\sqrt{3}/12)a_1 d_{31}
-(1/4)a_1 d_{32}-r_{1z})^2 \tag{50}
$$

$$
\ell_{11}^2=((\sqrt{3}/3)a_2 d^*_{11}+ r_x+ (\sqrt{3}/3)a_1 d_{11}-r_{1x})^2
+((\sqrt{3}/3)a_2 d^*_{21}+ r_y+ (\sqrt{3}/3)a_1 d_{21}-r_y)^2
+((\sqrt{3}/3)a_2 d^*_{31}+ r_z+ (\sqrt{3}/3)a_1 d_{31}-r_{1z})^2 \tag{51}
$$

$$
\ell_{12}^2=((\sqrt{3}/3)a_2 d^*_{11}+ r_x- (\sqrt{3}/6)a_1 d_{11}-r_{1x})
+((\sqrt{3}/3)a_2 d^*_{21}+ r_y- (\sqrt{3}/6)a_1 d_{21}-r_{1y})^2
+((\sqrt{3}/3)a_2 d^*_{31}+ r_z- (\sqrt{3}/6)a_1 d_{31}-r_{1z})^2 \tag{52}
$$

Note that the transformations T_o, T_1 and T_2 can also be expressed in terms of the associated Euler's angles θ, φ and ψ such that

$$
d_{11}= C\theta,\quad d_{12}= C\theta\, S\varphi\, S\psi - S\theta\, C\psi \tag{53}
$$

$$
d_{13}= C\theta\, S\varphi\, C\psi + S\varphi\, S\psi,\quad d_{21}= S\varphi\, C\varphi \tag{54}
$$

$$
d_{22}= S\theta\, S\varphi\, S\psi + C\theta\, C\psi \tag{55}
$$

$$d_{23} = S\theta\, S\varphi\, C\psi - C\varphi\, S\psi \tag{56}$$

$$d_{31} = S\varphi,\ d_{32} = C\varphi\, S\psi,\ d_{33} = C\varphi\, C\psi. \tag{57}$$

where the symbols C and S stand for Cosine and Sine of an angle.
Thus, all d_{ij}'s can be expressed in terms of three Euler's angle θ, φ and ψ [see chapter 2 of Shahinpoor [1]].

Now given the position and the orientation of the frame T_2 with respect to the reference base frame T_o it is true that

$$T_o^2 = T_o^1\, T_1^2, \tag{58}$$

where T_j^i is the 4x4 homogeneous transformation describing the position and the orientations of frame T_i with respect to frame T_j. In terms of the Euler's angles θ, φ, ψ, and θ_2, φ_2, ψ_2 and the position vectors $\mathbf{r}_1 = (r_{1x}, r_{1y}, r_{1z})$ and $\mathbf{r}_2 = (r_{2x}, r_{2y}, r_{2z})$, with respect to T_o and T_1 frames, respectively, the following relationships hold true

$$\textbf{Euler}\ (\theta_1, \varphi_1, \psi_1, r_{1x}, r_{1y}, r_{1z})\ \textbf{Euler}\ (\theta_2, \varphi_2, \psi_2, r_{2x}, r_{2y}, r_{2z}) = \textbf{Euler}\ (\theta, \varphi, \psi, r_x, r_y, r_z), \tag{59}$$

where

$$T_o^1 = \textbf{Euler}\ (\theta_1, \varphi_1, \psi_1, r_{1x}, r_{1y}, r_{1z}) \tag{60}$$

$$T_1^2 = \textbf{Euler}\ (\theta_2, \varphi_2, \psi_2, r_{2x}, r_{2y}, r_{2z}) \tag{61}$$

$$T_o^2 = \textbf{Euler}\ (\theta, \varphi, \psi, r_x, r_y, r_z). \tag{62}$$

Now given T_o^2 in order to find the 6 actuation length ℓ_1, ℓ_3, ℓ_5, ℓ_8, ℓ_{10} and ℓ_{12} in terms of the known geometrical quantities ℓ_2, ℓ_4, ℓ_6, ℓ_7, ℓ_9, ℓ_{11}, a_o, a_1, a_2, one must solve 24 equations with 18 unknowns. The unknowns are θ_1, φ_1, ψ_1, r_{1x}, r_{1y}, r_{1z}, θ_2, φ_2, ψ_2, r_{2x}, r_{2y}, r_{2z}, ℓ_1, ℓ_3, ℓ_5, ℓ_8, ℓ_{10} and ℓ_{12}.
Note that under these circumstances

$$\textbf{Euler}\ (\theta_1, \varphi_1, \psi_1, r_{1x}, r_{1y}, r_{1z}) = \begin{bmatrix} C\theta_1 & C\theta_1 S\varphi_1 S\psi_1 - S\theta_1 C\psi_1 & C\theta_1 S\varphi_1 C\psi_1 + S\varphi_1 S\psi_1 & r_{1x} \\ S\varphi_1 C\varphi_1 & S\theta_1 S\varphi_1 S\psi_1 + C\theta_1 C\psi_1 & S\theta_1 S\varphi_1 C\psi_1 - C\varphi_1 S\psi_1 & r_{1y} \\ -S\varphi_1 & C\varphi_1 S\psi_1 & C\varphi_1 C\psi_1 & r_{1z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{63}$$

$$\textbf{Euler}\ (\theta_2, \varphi_2, \psi_2, r_{2x}, r_{2y}, r_{2z}) = \begin{bmatrix} C\theta_2 & C\theta_2 S\varphi_2 S\psi_2 - S\theta_2 C\psi_2 & C\theta_2 S\varphi_2 C\psi_2 + S\varphi_2 S\psi_2 & r_{2x} \\ S\varphi_2 C\varphi_2 & S\theta_2 S\varphi_2 S\psi_2 + C\theta_2 C\psi_2 & S\theta_2 S\varphi_2 C\psi_2 - C\varphi_2 S\psi_2 & r_{2y} \\ -S\varphi_2 & C\varphi_2 S\psi_2 & C\varphi_2 C\psi_2 & r_{2z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{64}$$

$$\textbf{Euler}\ (\theta, \varphi, \psi, r_x, r_y, r_z) = \begin{bmatrix} C\theta & C\theta S\phi S\Psi - S\theta C\Psi & C\theta S\phi C\Psi + S\phi S\Psi & r_x \\ S\phi C\phi & S\theta S\phi S\Psi + C\theta C\Psi & S\theta S\phi C\Psi - C\phi S\Psi & r_y \\ -S\phi & C\phi S\Psi & C\phi C\Psi & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{65}$$

Thus

11→ $$C\theta = C\theta_1 C\theta_2 + S\varphi_2 C\varphi_2 (C\varphi_1 S\varphi_1 S\psi_1 - S\theta_1 C\psi_1) - S\varphi(C\theta_1 S\varphi_1 C\psi_1 + S\varphi_1 S\psi_1) \tag{66}$$

12→ $$C\theta S\varphi S\psi - S\theta C\psi = C\theta_1(C\theta_2 S\varphi_2 S\psi_2 - S\theta_2 C\psi_2) + (C\theta_1 S\varphi_1 S\psi_1 - S\theta_1 C\psi_1)(S\theta_2 S\theta_2 S\psi_2 + C\theta_2 C\psi_2) + C\varphi_2 S\psi_2 (C\theta_1 S\varphi_1 C\psi_1 + S\varphi_1 S\psi_1) \tag{67}$$

13→ $$C\theta S\varphi C\psi + S\varphi S\psi = C\theta_1(C\theta_2 S\varphi_2 C\psi_2 + S\varphi_2 S\psi_2) + (C\theta_1 S\varphi_1 S\psi_1 - S\theta_1 C\psi_1)(S\theta_2 S\varphi_2 C\psi_2 - C\varphi_2 S\psi_2) + C\varphi_2 C\psi_2 (C\theta_1 S\theta_1 C\psi_1 + S\varphi_1 S\psi_1) \tag{68}$$

14→ $$r_x = r_{2x}\, C\theta_1 + r_{2y}(C\theta_1 S\varphi_1 S\psi_1 - S\theta_1 C\psi_1) + r_{2z}(C\theta_1 S\varphi_1 C\psi_1 + S\varphi_1 S\psi_1) + r_{1x} \tag{69}$$

24→ $$r_y = r_{2x}(S\varphi_1 C\varphi_1) + r_{2y}(S\theta_1 S\varphi_1 S\psi_1 + C\theta_1 C\psi_1)\ r_{2z}(S\theta_1 S\varphi_1 C\psi_1 - C\varphi_1 S\psi_1) + r_{1y} \tag{70}$$

34→ $$r_z = r_{2x}(-S\varphi_1) + r_{2y}(C\varphi_1 S\psi_1) + r_{2z}(C\varphi_1 C\psi_1) + r_{1z} \tag{71}$$

21→ $$S\varphi C\varphi = C\theta_2(S\varphi_1 C\varphi_1) + S\varphi_2 C\varphi_2(S\theta_1 S\varphi_1 S\psi_1 + C\theta_1 C\psi_1) - S\varphi(S\theta_1 S\varphi_1 C\psi_1 - C\theta_1 S\psi_1) \tag{72}$$

31→ $S\varphi = C\theta_2(-S\varphi_1)+S\varphi_2C\varphi_2(C\varphi_1S\psi_1)-S\varphi_2C\varphi_1C\psi_1$ (73)

32→ $C\varphi S\psi = -S\varphi_1(C\theta_2S\varphi_2S\psi_2 - S\theta_2C\psi_2) +C\varphi_1S\psi_1(S\theta_2S\varphi_2S\psi_2 + C\theta_2C\psi_2) +C\psi_1C\psi_1\ C\varphi_2S\psi_2$ (74)

33→ $C\varphi C\psi = -S\varphi_1(C\theta_2S\varphi_2C\psi_2 + S\varphi_2S\psi_2) +C\varphi_1S\psi_1(S\theta_2S\varphi_2C\psi_2 - C\varphi_2S\psi_2) +C\varphi_1C\psi_1C\varphi_2C\psi_2$ (75)

22→ $S\theta S\varphi S\psi + C\theta C\psi = S\varphi_1C\varphi_1(C\theta_2S\varphi_2S\psi_2 - S\theta_2C\psi_2)+(S\theta_1S\varphi_1S\psi_1 + C\theta_1C\psi_1)(S\theta_2S\varphi_2S\psi_2 +C\theta_2C\psi_2)C\varphi_2S\psi_2(S\theta_1S\varphi_1C\psi_1-C\varphi_1S\psi_1)$ (76)

23→ $S\theta S\varphi C\psi - C\varphi S\psi = S\varphi_1C\varphi_1(C\theta_2S\varphi_2C\psi_2 + S\varphi_2S\psi_2)+(S\theta_1S\varphi_1S\psi_1 + C\theta_1C\psi_1)(S\theta_2S\varphi_2C\psi_2 - C\varphi_2S\psi_2)+C\varphi_2C\psi_2(S\theta_1S\varphi_1C\psi_1 - C\varphi_1S\psi_1).$ (77)

In addition to the above equations the following equations are also true:

$$\ell_1^2 =[(\sqrt{3}/6)a_1C\theta_1-(1/2)a_1(C\theta_1S\varphi_1S\psi_1-S\theta_1C\psi_1) +r_{1x}- (\sqrt{3}/6)a_o)^2+[(\sqrt{3}/6)a_1S\varphi_1C\varphi_1 -(1/2)a_1(S\theta_1S\varphi_1S\psi_1+C\theta_1C\psi_1)+r_{1y}+(a_o/2)]^2 +[-(\sqrt{3}/6)a_1S\varphi_1-(1/2)a_1C\varphi_1S\psi_1+r_{1z}]]^2 \quad (78)$$

$$\ell_2^2=[-(\sqrt{3}/12)a_1C\theta_1+(1/4)a_1(C\theta_1S\varphi_1S\psi_1 -S\theta_1C\psi_1)+r_{1x}- (\sqrt{3}/6)a_o]^2+[-(\sqrt{3}/12)a_1S\varphi_1 +(1/4)a_1(S\theta_1S\varphi_1S\psi_1+C\theta_1C\psi_1)+r_{1y}+(a_o/2]^2 +[+ (\sqrt{3}/12)a_1S\varphi_1+(1/4)a_1C\varphi_1S\psi_1+r_{1z}]^2 \quad (79)$$

$$\ell_3^2 =[- (\sqrt{3}/3)a_1C\theta_1+r_{1x}+ (\sqrt{3}/3)a_o]^2+[-(\sqrt{3}/3) a_1S\varphi_1C\varphi_1+r_{1y}]^2+[+(\sqrt{3}/3)a_1S\varphi_1+r_{1z}]^2 \quad (80)$$

$$\ell_4^2 =[(\sqrt{3}/6)a_1C\theta_1+r_{1x}+ (\sqrt{3}/3)a_o]^2 +[(\sqrt{3}/6)a_1S\varphi_1C\varphi_1+r_{1y}]^2 +[- (\sqrt{3}/6)a_1S\varphi_1+r_{1z}]^2 \quad (81)$$

$$\ell_5^2=[(\sqrt{3}/6)a_1C\theta_1+(1/2)a_1(C\theta_1S\varphi_1S\psi_1 -S\theta_1C\psi_1)+r_{1x}- (\sqrt{3}/6)a_o]^2+[+(\sqrt{3}/6)a_1S\varphi_1C\varphi_1 +(1/2)a_1(S\theta_1S\varphi_1S\psi_1+C\theta_1C\psi_1)+r_{1y}-(1/2)a_o]^2 +[- (\sqrt{3}/6)a_1S\varphi_1+(1/2)a_1C\varphi_1S\psi_1+r_{1z}]^2 \quad (82)$$

$$\ell_6^2=[-(\sqrt{3}/12)a_1C\theta_1-(1/4)a_1(C\theta_1S\varphi_1S\psi_1 -S\theta_1C\psi_1)+r_{1x}-(\sqrt{3}/6)a_o]^2+[-(\sqrt{1}3/12)a_1S\varphi_1C\varphi_1-(1/4)a_1(S\theta_1S\varphi_1S\psi_1 +C\theta_1C\psi_1)+r_{1y}-(1/2)a_o]^2+[\sqrt{3}/12)a_1S\varphi_1 -(1/4)a_1C\varphi_1S\varphi_1+r_{1z}]^2 -(1/2)a_2C\varphi_2S\psi_2+r_y-(\sqrt{3}/6)a_1S\varphi_1C\varphi_1 -(1/2)a_1(S\theta_1S\varphi_1S\psi_1+C\theta_1C\psi\ )-r_{1y}]^2 \quad (83)$$

$$\ell_7^2=[(\sqrt{3}/6)a_2C\theta_2-(1/2)a_2(C\theta_2S\varphi_2S\psi_2-S\theta_2C\psi_2) +r_x-(\sqrt{3}/6)a_1C\theta_1-(1/2)a_1(C\theta_1S\varphi_1S\psi_1 -S\theta_1C\psi_1)-r_{1x}]^2+[(\sqrt{3}/6)a_2S\varphi_2C\varphi_2 -(1/2)a_2C\varphi_2S\psi_2+r_y-(\sqrt{3}/6)a_1S\varphi_1C\varphi_1 -(1/2)a_1(S\theta_1S\varphi_1S\psi_1+C\theta_1C\psi_1)-R_{1y}]^2 +[-(\sqrt{3}/6)a_2S\varphi_2-(1/2)a_2(S\theta_2S\varphi_2S\psi_2 +C\theta_2C\psi_2)+r_z+(\sqrt{3}/6)a_1S\varphi_1 -(1/2)a_1C(_1S\psi_1-r_{1z}]^2 \quad (84)$$

Similar expressions follow for ℓ_8^2 through ℓ_{12}^2.

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